Independent Study of General Relativity

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2026-01-09

Contents

Introduction

This is my personal record of studying general relativity. I am writing it to:

- organize the material in a way that makes sense to me
- work through the derivations step by step
- force myself to understand the material more deeply by writing it out, and
- create a reference I can return to as I continue learning.

This is not intended as a textbook or a guide for others. It's simply my personal working notes. It is still a work in progress. Many sections are incomplete, and others will change as I learn more and expand my knowledge.

Acknowledgments

In my years of schooling I had many math and science teachers. Even though I can no longer remember most of their names, I appreciate all that they taught me. With respect to calculus and physics, there were two teachers at Mountain View High School in Mesa, AZ who helped spark my interest in these subjects:

- Mr. Rex Rice — AP Physics
- Mr. Wayne Slade — Trigonometry & Math Analysis, and AP Calculus

The EigenChris YouTube channel has been an important learning resource. I have tried to study this topic in the past and was never able to get a grasp of tensors from just reading textbooks. Its three playlists *Tensors for Beginners*, *Tensor Calculus*, and *Relativity by EigenChris* have been especially helpful.

ChatGPT has provided on-demand explanations, clarifications, and guidance while I worked through the mathematics and physics of general relativity.

I am likewise grateful for the many scientists and mathematicians whose work forms the foundation of general relativity, including Albert Einstein, Karl Schwarzschild, Bernhard Riemann, Elwin Christoffel, Tullio Levi-Civita, and Hermann Minkowski.

PDF: BradGarn.com/GR.pdf
Source: github.com/bradhg/GR
Git SHA: 360aacf

1. The Einstein Field Equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Sybmbol	Name	Quantity	Dimensions
$R_{\mu\nu}$	Ricci tensor	curvature	L^{-2}
$g_{\mu\nu}$	Metric tensor	dimensionless	—
R	Ricci scalar	curvature	L^{-2}
G	Universal constant	gravitational coupling	$L^3 M^{-1} T^{-2}$
c	Speed of light	speed	$L T^{-1}$
$T_{\mu\nu}$	Stress–energy tensor	energy density	$M L^{-1} T^{-2}$

Table 1: Key quantities in general relativity

1.1. Stress–energy tensor $T_{\mu\nu}$

The stress–energy tensor describes how energy and momentum are spread out in space and how they flow. It bundles together rest–mass ρc^2, thermal, kinetic, and radiation contributions, varying from point to point in spacetime. In a chosen frame, T^{00} is the energy density, T^{0i} gives the flow of energy (or momentum density), and T^{ij} gives the stresses such as pressure and shear.

1.2. The Metric Tensor

$$\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^{\mu}\, \mathrm{d}x^{\nu}$$

1.3. The Christoffel Symbols

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu} \right)$$

1.4. The Riemann tensor

$$R^{\alpha}_{\mu\beta\nu} = \partial_{\beta} \Gamma^{\alpha}_{\mu\nu} - \partial_{\nu} \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\nu} \Gamma^{\rho}_{\beta\rho} - \Gamma^{\alpha}_{\mu\rho} \Gamma^{\rho}_{\beta\nu}$$

1.5. The Ricci tensor

As a contraction of the Riemann tensor

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$$

Or directly from the Christoffel symbols

$$R_{\mu\nu} = \partial_{\lambda} \Gamma^{\lambda}_{\mu\nu} - \partial_{\nu} \Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\lambda}_{\mu\nu} \Gamma^{\rho}_{\lambda\rho} - \Gamma^{\lambda}_{\mu\rho} \Gamma^{\rho}_{\lambda\nu}$$

1.6. The Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu}$$

2. The Schwarzschild Metric Derivation

2.1. The metric and it inverse

The Schwarzschild Metric is for the empty space near a spherically symmetric non-rotating chargeless mass. The coordinates are $x^{\mu} = (ct, r, \theta, \varphi)$. We will begin with the standard ansatz for the Schwarzschild metric where A and B are unknow functions of r. The other two non-zero terms in the metric are determined by requiring spherical symmetry. Being static makes all g_{ti} and g_{it} terms equal zero. Spherical symmetry requires all $g_{r\theta}$, $g_{r\varphi}$, and $g_{\theta\varphi}$ terms to be zero.

$$g_{\mu\nu} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

Because the metric is diagonal the inverse metric is simply the reciprocal of each element.

$$g^{\mu\nu} = \begin{pmatrix} \frac{1}{A} & 0 & 0 & 0 \\ 0 & \frac{1}{B} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{pmatrix}$$

2.2. The Partial Derivatives

$$\begin{aligned} \partial_r g_{tt} &= A' \\ \partial_r g_{rr} &= B' \\ \partial_r g_{\theta\theta} &= 2r \\ \partial_r g_{\varphi\varphi} &= 2r \sin^2\theta \\ \partial_\theta g_{\varphi\varphi} &= 2r^2 \sin\theta \cos\theta \\ \text{others} &= 0 \end{aligned}$$

2.3. The Christoffel Symbols

Because this metric is diagonal the Christoffel Symbols equations simplifies to the following:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right) \qquad \text{no sum on } \rho$$

With three indexes in 4d spacetime there are 64 Christoffel Symbols. But they are symmetric in the lower index so that leaves only 40. And because there are only 5 partial derivatives of the metric that are non-zero only the following Christoffel Symbols are non-zero.

$$
\begin{aligned}
\Gamma^t_{tr} = \Gamma^t_{rt} &= \frac{1}{2} g^{tt}\left(\cancel{\partial_t g_{rt}} + \partial_r g_{tt} - \cancel{\partial_t g_{tr}}\right) &&= \frac{A'}{2A} \\
\Gamma^r_{tt} &= \frac{1}{2} g^{rr}\left(\cancel{\partial_t g_{tr}} + \cancel{\partial_t g_{tr}} - \partial_r g_{tt}\right) &&= \frac{-A'}{2B} \\
\Gamma^r_{rr} &= \frac{1}{2} g^{rr}\left(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr}\right) &&= \frac{B'}{2B} \\
\Gamma^r_{\theta\theta} &= \frac{1}{2} g^{rr}\left(\cancel{\partial_\theta g_{\theta r}} + \cancel{\partial_\theta g_{\theta r}} - \partial_r g_{\theta\theta}\right) &&= \frac{-2r}{2B} = \frac{-r}{B} \\
\Gamma^r_{\varphi\varphi} &= \frac{1}{2} g^{rr}\left(\cancel{\partial_\varphi g_{\varphi r}} + \cancel{\partial_\varphi g_{\varphi r}} - \partial_r g_{\varphi\varphi}\right) &&= \frac{-2r\sin^2\theta}{2B} = \frac{-r\sin^2\theta}{B} \\
\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} &= \frac{1}{2} g^{\theta\theta}\left(\cancel{\partial_\theta g_{r\theta}} + \partial_r g_{\theta\theta} - \cancel{\partial_\theta g_{\theta r}}\right) &&= \frac{2r}{2r^2} = \frac{1}{r} \\
\Gamma^\theta_{\varphi\varphi} &= \frac{1}{2} g^{\theta\theta}\left(\cancel{\partial_\varphi g_{\varphi\theta}} + \cancel{\partial_\varphi g_{\varphi\theta}} - \partial_\theta g_{\varphi\varphi}\right) &&= \frac{-2r^2\sin\theta\cos\theta}{2r^2} = -\sin\theta\cos\theta \\
\Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} &= \frac{1}{2} g^{\varphi\varphi}\left(\cancel{\partial_\varphi g_{r\varphi}} + \partial_r g_{\varphi\varphi} - \cancel{\partial_\varphi g_{\varphi r}}\right) &&= \frac{2r\sin^2\theta}{2r^2\sin^2\theta} = \frac{1}{r} \\
\Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} &= \frac{1}{2} g^{\varphi\varphi}\left(\cancel{\partial_\varphi g_{\theta\varphi}} + \partial_\theta g_{\varphi\varphi} - \cancel{\partial_\varphi g_{\varphi\theta}}\right) &&= \frac{2r^2\sin\theta\cos\theta}{2r^2\sin^2\theta} = \cot\theta
\end{aligned}
$$

2.4. The Ricci Tensor

2.4.1. R_{tt} Component of the Ricci Tensor

$$
R_{tt} = \partial_\lambda \Gamma^\lambda_{tt} - \partial_t \Gamma^\lambda_{t\lambda} + \Gamma^\lambda_{tt}\Gamma^\rho_{\lambda\rho} - \Gamma^\lambda_{t\rho}\Gamma^\rho_{\lambda t}
$$

R_{tt} **First Term** Only the partial with respect to r will be non-zero

$$
\begin{aligned}
\partial_\lambda \Gamma^\lambda_{tt} &= \cancel{\partial_t \Gamma^t_{tt}} + \partial_r \Gamma^r_{tt} + \cancel{\partial_\theta \Gamma^\theta_{tt}} + \cancel{\partial_\varphi \Gamma^\varphi_{tt}} \\
&= \partial_r \frac{-A'}{2B} \\
&= \frac{(-2B')(-A') + (-A'')(2B)}{4B^2} \\
&= \frac{A'B'}{2B^2} - \frac{A''}{2B}
\end{aligned}
$$

R_{tt} **Second Term** All partials with respect to t are zero.

$$
\begin{aligned}
\partial_t \Gamma^\lambda_{t\lambda} &= \cancel{\partial_t \Gamma^t_{tt}} + \cancel{\partial_t \Gamma^r_{tr}} + \cancel{\partial_t \Gamma^\theta_{t\theta}} + \cancel{\partial_t \Gamma^\varphi_{t\varphi}} \\
&= 0
\end{aligned}
$$

R_{tt} **Third Term** The only value of λ that has a non-zero Christoffel is r and then all values of ρ will have non-zero Christoffel Symbols.

$$\begin{aligned}
\Gamma^{\lambda}_{tt}\Gamma^{\rho}_{\lambda\rho} = \; & \cancel{\Gamma^{t}_{tt}\Gamma^{t}_{tt}} + \cancel{\Gamma^{t}_{tt}\Gamma^{r}_{tr}} + \cancel{\Gamma^{t}_{tt}\Gamma^{\theta}_{t\theta}} + \cancel{\Gamma^{t}_{tt}\Gamma^{\varphi}_{t\varphi}} \\
& + \Gamma^{r}_{tt}\Gamma^{t}_{rt} + \Gamma^{r}_{tt}\Gamma^{r}_{rr} + \Gamma^{r}_{tt}\Gamma^{\theta}_{r\theta} + \Gamma^{r}_{tt}\Gamma^{\varphi}_{r\varphi} \\
& + \cancel{\Gamma^{\theta}_{tt}\Gamma^{t}_{\theta t}} + \cancel{\Gamma^{\theta}_{tt}\Gamma^{r}_{\theta r}} + \cancel{\Gamma^{\theta}_{tt}\Gamma^{\theta}_{\theta\theta}} + \cancel{\Gamma^{\theta}_{tt}\Gamma^{\varphi}_{\theta\varphi}} \\
& + \cancel{\Gamma^{\varphi}_{tt}\Gamma^{t}_{\varphi t}} + \cancel{\Gamma^{\varphi}_{tt}\Gamma^{r}_{\varphi r}} + \cancel{\Gamma^{\varphi}_{tt}\Gamma^{\theta}_{\varphi\theta}} + \cancel{\Gamma^{\varphi}_{tt}\Gamma^{\varphi}_{\varphi\varphi}} \\
= \; & \frac{-A'}{2B}\frac{A'}{2A} + \frac{-A'}{2B}\frac{B'}{2B} + \frac{-A'}{2B}\frac{1}{r} + \frac{-A'}{2B}\frac{1}{r} \\
= \; & -\frac{A'^2}{4AB} - \frac{A'B'}{4B^2} - \frac{A'}{rB}
\end{aligned}$$

R_{tt} **Fourth Term** The only non-zero Christoffel symbols have one r and two t indices.

$$\begin{aligned}
\Gamma^{\lambda}_{t\rho}\Gamma^{\rho}_{\lambda t} = \; & \cancel{\Gamma^{t}_{tt}\Gamma^{t}_{tt}} + \Gamma^{t}_{tr}\Gamma^{r}_{tt} + \cancel{\Gamma^{t}_{t\theta}\Gamma^{\theta}_{tt}} + \cancel{\Gamma^{t}_{t\varphi}\Gamma^{\varphi}_{tt}} \\
& + \Gamma^{r}_{tt}\Gamma^{t}_{rt} + \cancel{\Gamma^{r}_{tr}\Gamma^{r}_{rt}} + \cancel{\Gamma^{r}_{t\theta}\Gamma^{\theta}_{rt}} + \cancel{\Gamma^{r}_{t\varphi}\Gamma^{\varphi}_{rt}} \\
& + \cancel{\Gamma^{\theta}_{tt}\Gamma^{t}_{\theta t}} + \cancel{\Gamma^{\theta}_{tr}\Gamma^{r}_{\theta t}} + \cancel{\Gamma^{\theta}_{t\theta}\Gamma^{\theta}_{\theta t}} + \cancel{\Gamma^{\theta}_{t\varphi}\Gamma^{\varphi}_{\theta t}} \\
& + \cancel{\Gamma^{\varphi}_{tt}\Gamma^{t}_{\varphi t}} + \cancel{\Gamma^{\varphi}_{tr}\Gamma^{r}_{\varphi t}} + \cancel{\Gamma^{\varphi}_{t\theta}\Gamma^{\theta}_{\varphi t}} + \cancel{\Gamma^{\varphi}_{t\varphi}\Gamma^{\varphi}_{\varphi t}} \\
= \; & \frac{-A'}{2B}\frac{A'}{2A} + \frac{A'}{2A}\frac{-A'}{2B} \\
= \; & -\frac{A'^2}{2AB}
\end{aligned}$$

R_{tt} Complete

$$R_{tt} = \frac{A'B'}{2B^2} - \frac{A''}{2B} - 0 + \frac{-A'^2}{4AB} - \frac{A'B'}{4B^2} + \frac{-A'}{rB} - \frac{-A'^2}{2AB}$$

$$R_{tt} = -\frac{A''}{2B} + \frac{A'B'}{4B^2} + \frac{A'^2}{4AB} - \frac{A'}{rB} \tag{2.1}$$

2.4.2. R_{rr} Component of the Ricci Tensor

$$R_{rr} = \partial_{\lambda}\Gamma^{\lambda}_{rr} - \partial_r\Gamma^{\lambda}_{r\lambda} + \Gamma^{\lambda}_{rr}\Gamma^{\rho}_{\lambda\rho} - \Gamma^{\lambda}_{r\rho}\Gamma^{\rho}_{\lambda r}$$

R_{rr} **First Term** Only the partial with respect to r will be non-zero

$$\begin{aligned}
\partial_{\lambda}\Gamma^{\lambda}_{rr} &= \cancel{\partial_t\Gamma^{t}_{rr}} + \partial_r\Gamma^{r}_{rr} + \cancel{\partial_\theta\Gamma^{\theta}_{rr}} + \cancel{\partial_\varphi\Gamma^{\varphi}_{rr}} \\
&= \partial_r\frac{B'}{2B} \\
&= \frac{(-2B')(B') + (B'')(2B)}{4B^2} \\
&= -\frac{B'^2}{2B^2} + \frac{B''}{2B}
\end{aligned}$$

R_{rr} **Second Term**

$$
\begin{aligned}
\partial_r \Gamma^{\lambda}_{r\lambda} &= \partial_r \Gamma^{t}_{rt} + \partial_r \Gamma^{r}_{rr} + \partial_r \Gamma^{\theta}_{r\theta} + \partial_r \Gamma^{\varphi}_{r\varphi} \\
&= \partial_r \frac{A'}{2A} + \partial_r \frac{B'}{2B} + \partial_r \frac{1}{r} + \partial_r \frac{1}{r} \\
&= \frac{(-2A')(A') + (A'')(2A)}{4A^2} + \frac{(-2B')(B') + (B'')(2B)}{4B^2} + \frac{-1}{r^2} + \frac{-1}{r^2} \\
&= \frac{-A'^2}{2A^2} + \frac{A''}{2A} + \frac{-B'^2}{2B^2} + \frac{B''}{2B} + \frac{-2}{r^2} \\
&= -\frac{A'^2}{2A^2} + \frac{A''}{2A} - \frac{B'^2}{2B^2} + \frac{B''}{2B} - \frac{2}{r^2}
\end{aligned}
$$

R_{rr} **Third Term** The only value of λ that has a non-zero Christoffel is r and then all values of ρ will have non-zero Christoffel Symbols.

$$
\begin{aligned}
\Gamma^{\lambda}_{rr}\Gamma^{\rho}_{\lambda\rho} = \; & \cancel{\Gamma^{t}_{rr}\Gamma^{t}_{tt}} + \cancel{\Gamma^{t}_{rr}\Gamma^{r}_{tr}} + \cancel{\Gamma^{t}_{rr}\Gamma^{\theta}_{t\theta}} + \cancel{\Gamma^{t}_{rr}\Gamma^{\varphi}_{t\varphi}} \\
& + \Gamma^{r}_{rr}\Gamma^{t}_{rt} + \Gamma^{r}_{rr}\Gamma^{r}_{rr} + \Gamma^{r}_{rr}\Gamma^{\theta}_{r\theta} + \Gamma^{r}_{rr}\Gamma^{\varphi}_{r\varphi} \\
& + \cancel{\Gamma^{\theta}_{rr}\Gamma^{t}_{\theta t}} + \cancel{\Gamma^{\theta}_{rr}\Gamma^{r}_{\theta r}} + \cancel{\Gamma^{\theta}_{rr}\Gamma^{\theta}_{\theta\theta}} + \cancel{\Gamma^{\theta}_{rr}\Gamma^{\varphi}_{\theta\varphi}} \\
& + \cancel{\Gamma^{\varphi}_{rr}\Gamma^{t}_{\varphi t}} + \cancel{\Gamma^{\varphi}_{rr}\Gamma^{r}_{\varphi r}} + \cancel{\Gamma^{\varphi}_{rr}\Gamma^{\theta}_{\varphi\theta}} + \cancel{\Gamma^{\varphi}_{rr}\Gamma^{\varphi}_{\varphi\varphi}} \\
= \; & \frac{B'}{2B}\frac{A'}{2A} + \frac{B'}{2B}\frac{B'}{2B} + \frac{B'}{2B}\frac{1}{r} + \frac{B'}{2B}\frac{1}{r} \\
= \; & \frac{A'B'}{4AB} + \frac{B'^2}{4B^2} + \frac{B'}{rB}
\end{aligned}
$$

R_{rr} **Fourth Term** The only non-zero Christoffel symbols have one r index and the other two indices are equal.

$$
\begin{aligned}
\Gamma^{\lambda}_{r\rho}\Gamma^{\rho}_{\lambda r} = \; & \Gamma^{t}_{rt}\Gamma^{t}_{tr} + \cancel{\Gamma^{t}_{rr}\Gamma^{r}_{tr}} + \cancel{\Gamma^{t}_{r\theta}\Gamma^{\theta}_{tr}} + \cancel{\Gamma^{t}_{r\varphi}\Gamma^{\varphi}_{tr}} \\
& + \cancel{\Gamma^{r}_{rt}\Gamma^{t}_{rr}} + \Gamma^{r}_{rr}\Gamma^{r}_{rr} + \cancel{\Gamma^{r}_{r\theta}\Gamma^{\theta}_{rr}} + \cancel{\Gamma^{r}_{r\varphi}\Gamma^{\varphi}_{rr}} \\
& + \cancel{\Gamma^{\theta}_{rt}\Gamma^{t}_{\theta r}} + \cancel{\Gamma^{\theta}_{rr}\Gamma^{r}_{\theta r}} + \Gamma^{\theta}_{r\theta}\Gamma^{\theta}_{\theta r} + \cancel{\Gamma^{\theta}_{r\varphi}\Gamma^{\varphi}_{\theta r}} \\
& + \cancel{\Gamma^{\varphi}_{rt}\Gamma^{t}_{\varphi r}} + \cancel{\Gamma^{\varphi}_{rr}\Gamma^{r}_{\varphi r}} + \cancel{\Gamma^{\varphi}_{r\theta}\Gamma^{\theta}_{\varphi r}} + \Gamma^{\varphi}_{r\varphi}\Gamma^{\varphi}_{\varphi r} \\
= \; & \frac{A'}{2A}\frac{A'}{2A} + \frac{B'}{2B}\frac{B'}{2B} + \frac{1}{r}\frac{1}{r} + \frac{1}{r}\frac{1}{r} \\
= \; & \frac{A'^2}{4A^2} + \frac{B'^2}{4B^2} + \frac{2}{r^2}
\end{aligned}
$$

R_{rr} **Complete**

$$R_{rr} = \left(-\frac{B'^2}{2B^2} + \frac{B''}{2B}\right)$$
$$-\left(-\frac{A'^2}{2A^2} + \frac{A''}{2A} - \frac{B'^2}{2B^2} + \frac{B''}{2B} - \frac{2}{r^2}\right)$$
$$+\left(\frac{A'B'}{4AB} + \frac{B'^2}{4B^2} + \frac{B'}{rB}\right)$$
$$-\left(\frac{A'^2}{4A^2} + \frac{B'^2}{4B^2} + \frac{2}{r^2}\right)$$

$$R_{rr} = -\frac{A''}{2A} + \frac{A'B'}{4AB} + \frac{A'^2}{4A^2} + \frac{B'}{rB} \tag{2.2}$$

2.4.3. $R_{\theta\theta}$ Component of the Ricci Tensor

$$R_{\theta\theta} = \partial_\lambda \Gamma^\lambda_{\theta\theta} - \partial_\theta \Gamma^\lambda_{\theta\lambda} + \Gamma^\lambda_{\theta\theta}\Gamma^\rho_{\lambda\rho} - \Gamma^\lambda_{\theta\rho}\Gamma^\rho_{\lambda\theta}$$

$R_{\theta\theta}$ **First Term** Only the partial with respect to r will be non-zero

$$\begin{aligned}\partial_\lambda \Gamma^\lambda_{\theta\theta} &= \cancel{\partial_t \Gamma^t_{\theta\theta}} + \partial_r \Gamma^r_{\theta\theta} + \cancel{\partial_\theta \Gamma^\theta_{\theta\theta}} + \cancel{\partial_\varphi \Gamma^\varphi_{\theta\theta}} \\ &= \partial_r \frac{-r}{B} \\ &= \frac{(-B')(-r) + (-1)(B)}{B^2} \\ &= \frac{rB'}{B^2} - \frac{1}{B}\end{aligned}$$

$R_{\theta\theta}$ **Second Term** Only the partial with respect to θ is non-zero

$$\begin{aligned}\partial_\theta \Gamma^\lambda_{\theta\lambda} &= \cancel{\partial_\theta \Gamma^t_{\theta t}} + \cancel{\partial_\theta \Gamma^r_{\theta r}} + \cancel{\partial_\theta \Gamma^\theta_{\theta\theta}} + \partial_\theta \Gamma^\varphi_{\theta\varphi} \\ &= \partial_\theta \cot\theta \\ &= -\frac{1}{\sin^2}\theta \\ &= -\csc^2\theta\end{aligned}$$

$R_{\theta\theta}$ **Third Term** The only value of λ that has a non-zero Christoffel is r and then all values of ρ will have non-zero Christoffel Symbols.

$$\begin{aligned}
\Gamma^{\lambda}_{\theta\theta}\Gamma^{\rho}_{\lambda\rho} = & \cancel{\Gamma^{t}_{\theta\theta}\Gamma^{t}_{tt}} + \cancel{\Gamma^{t}_{\theta\theta}\Gamma^{r}_{tr}} + \cancel{\Gamma^{t}_{\theta\theta}\Gamma^{\theta}_{t\theta}} + \cancel{\Gamma^{t}_{\theta\theta}\Gamma^{\varphi}_{t\varphi}} \\
& + \Gamma^{r}_{\theta\theta}\Gamma^{t}_{rt} + \Gamma^{r}_{\theta\theta}\Gamma^{r}_{rr} + \Gamma^{r}_{\theta\theta}\Gamma^{\theta}_{r\theta} + \Gamma^{r}_{\theta\theta}\Gamma^{\varphi}_{r\varphi} \\
& + \cancel{\Gamma^{\theta}_{\theta\theta}\Gamma^{t}_{\theta t}} + \cancel{\Gamma^{\theta}_{\theta\theta}\Gamma^{r}_{\theta r}} + \cancel{\Gamma^{\theta}_{\theta\theta}\Gamma^{\theta}_{\theta\theta}} + \cancel{\Gamma^{\theta}_{\theta\theta}\Gamma^{\varphi}_{\theta\varphi}} \\
& + \cancel{\Gamma^{\varphi}_{\theta\theta}\Gamma^{t}_{\varphi t}} + \cancel{\Gamma^{\varphi}_{\theta\theta}\Gamma^{r}_{\varphi r}} + \cancel{\Gamma^{\varphi}_{\theta\theta}\Gamma^{\theta}_{\varphi\theta}} + \cancel{\Gamma^{\varphi}_{\theta\theta}\Gamma^{\varphi}_{\varphi\varphi}} \\
= & \frac{-r}{B}\frac{A'}{2A} + \frac{-r}{B}\frac{B'}{2B} + \frac{-r}{B}\frac{1}{r} + \frac{-r}{B}\frac{1}{r} \\
= & -\frac{rA'}{2AB} - \frac{rB'}{2B^2} - \frac{2}{B}
\end{aligned}$$

$R_{\theta\theta}$ **Fourth Term** The only non-zero Christoffel symbols have one r and two angular indices.

$$\begin{aligned}
\Gamma^{\lambda}_{\theta\rho}\Gamma^{\rho}_{\lambda\theta} = & \cancel{\Gamma^{t}_{\theta t}\Gamma^{t}_{t\theta}} + \cancel{\Gamma^{t}_{\theta r}\Gamma^{r}_{t\theta}} + \cancel{\Gamma^{t}_{\theta\theta}\Gamma^{\theta}_{t\theta}} + \cancel{\Gamma^{t}_{\theta\varphi}\Gamma^{\varphi}_{t\theta}} \\
& + \cancel{\Gamma^{r}_{\theta t}\Gamma^{t}_{r\theta}} + \cancel{\Gamma^{r}_{\theta r}\Gamma^{r}_{r\theta}} + \Gamma^{r}_{\theta\theta}\Gamma^{\theta}_{r\theta} + \cancel{\Gamma^{r}_{\theta\varphi}\Gamma^{\varphi}_{r\theta}} \\
& + \cancel{\Gamma^{\theta}_{\theta t}\Gamma^{t}_{\theta\theta}} + \Gamma^{\theta}_{\theta r}\Gamma^{r}_{\theta\theta} + \cancel{\Gamma^{\theta}_{\theta\theta}\Gamma^{\theta}_{\theta\theta}} + \cancel{\Gamma^{\theta}_{\theta\varphi}\Gamma^{\varphi}_{\theta\theta}} \\
& + \cancel{\Gamma^{\varphi}_{\theta t}\Gamma^{t}_{\varphi\theta}} + \cancel{\Gamma^{\varphi}_{\theta r}\Gamma^{r}_{\varphi\theta}} + \cancel{\Gamma^{\varphi}_{\theta\theta}\Gamma^{\theta}_{\varphi\theta}} + \Gamma^{\varphi}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\theta} \\
= & \frac{-r}{B}\frac{1}{r} + \frac{1}{r}\frac{-r}{B} + \cot\theta\cot\theta \\
= & -\frac{2}{B} + \cot^2\theta
\end{aligned}$$

$R_{\theta\theta}$ Complete

$$\begin{aligned}
R_{\theta\theta} = & \frac{rB'}{B^2} - \frac{1}{B} \\
& -\left(-\csc^2(\theta)\right) \\
& +\left(-\frac{rA'}{2AB} - \frac{rB'}{2B^2} - \frac{2}{B}\right) \\
& -\left(-\frac{2}{B} + \cot^2\theta\right) \\
= & -\frac{rA'}{2AB} + \frac{rB'}{2B^2} - \frac{1}{B} + \csc^2\theta - \cot^2\theta
\end{aligned}$$

$$R_{\theta\theta} = -\frac{rA'}{2AB} + \frac{rB'}{2B^2} - \frac{1}{B} + 1 \tag{2.3}$$

2.4.4. $R_{\varphi\varphi}$ Component of the Ricci Tensor

$$R_{\varphi\varphi} = \partial_{\lambda}\Gamma^{\lambda}_{\varphi\varphi} - \partial_{\varphi}\Gamma^{\lambda}_{\varphi\lambda} + \Gamma^{\lambda}_{\varphi\varphi}\Gamma^{\rho}_{\lambda\rho} - \Gamma^{\lambda}_{\varphi\rho}\Gamma^{\rho}_{\lambda\varphi}$$

$R_{\varphi\varphi}$ **First Term** Only the partials with respect to r and θ will be non-zero

$$
\begin{aligned}
\partial_{\lambda}\Gamma^{\lambda}_{\varphi\varphi} &= \cancel{\partial_{t}\Gamma^{t}_{\varphi\varphi}} + \partial_{r}\Gamma^{r}_{\varphi\varphi} + \partial_{\theta}\Gamma^{\theta}_{\varphi\varphi} + \cancel{\partial_{\varphi}\Gamma^{\varphi}_{\varphi\varphi}} \\
&= \partial_r\left(\frac{-r\sin^2\theta}{B}\right) + \partial_\theta(-\sin\theta\cos\theta) \\
&= \frac{(B)\left(-\sin^2\theta\right) - \left(-r\sin^2\theta\right)(B')}{B^2} + (-\sin\theta)(-\sin\theta) + (\cos\theta)(-\cos\theta) \\
&= -\frac{\sin^2\theta}{B} + \frac{r\sin^2\theta B'}{B^2} + \sin^2\theta - \cos^2\theta
\end{aligned}
$$

$R_{\varphi\varphi}$ **Second Term** All partials with respect to φ are zero.

$$
\begin{aligned}
\partial_{\varphi}\Gamma^{\lambda}_{\varphi\lambda} &= \cancel{\partial_{\varphi}\Gamma^{t}_{\varphi t}} + \cancel{\partial_{\varphi}\Gamma^{r}_{\varphi r}} + \cancel{\partial_{\varphi}\Gamma^{\theta}_{\varphi\theta}} + \cancel{\partial_{\varphi}\Gamma^{\varphi}_{\varphi\varphi}} \\
&= 0
\end{aligned}
$$

$R_{\varphi\varphi}$ **Third Term** The only value of λ that has a non-zero Christoffel is r and then all values of ρ contribute.

$$
\begin{aligned}
\Gamma^{\lambda}_{\varphi\varphi}\Gamma^{\rho}_{\lambda\rho} =\ & \cancel{\Gamma^{t}_{\varphi\varphi}\Gamma^{t}_{tt}} + \cancel{\Gamma^{t}_{\varphi\varphi}\Gamma^{r}_{tr}} + \cancel{\Gamma^{t}_{\varphi\varphi}\Gamma^{\theta}_{t\theta}} + \cancel{\Gamma^{t}_{\varphi\varphi}\Gamma^{\varphi}_{t\varphi}} \\
& + \Gamma^{r}_{\varphi\varphi}\Gamma^{t}_{rt} + \Gamma^{r}_{\varphi\varphi}\Gamma^{r}_{rr} + \Gamma^{r}_{\varphi\varphi}\Gamma^{\theta}_{r\theta} + \Gamma^{r}_{\varphi\varphi}\Gamma^{\varphi}_{r\varphi} \\
& + \cancel{\Gamma^{\theta}_{\varphi\varphi}\Gamma^{t}_{\theta t}} + \cancel{\Gamma^{\theta}_{\varphi\varphi}\Gamma^{r}_{\theta r}} + \cancel{\Gamma^{\theta}_{\varphi\varphi}\Gamma^{\theta}_{\theta\theta}} + \Gamma^{\theta}_{\varphi\varphi}\Gamma^{\varphi}_{\theta\varphi} \\
& + \cancel{\Gamma^{\varphi}_{\varphi\varphi}\Gamma^{t}_{\varphi t}} + \cancel{\Gamma^{\varphi}_{\varphi\varphi}\Gamma^{r}_{\varphi r}} + \cancel{\Gamma^{\varphi}_{\varphi\varphi}\Gamma^{\theta}_{\varphi\theta}} + \cancel{\Gamma^{\varphi}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\varphi}} \\
=\ & \left(-\frac{r\sin^2\theta}{B}\right)\left(\frac{A'}{2A}\right) + \left(-\frac{r\sin^2\theta}{B}\right)\left(\frac{B'}{2B}\right) + \left(-\frac{r\sin^2\theta}{B}\right)\left(\frac{1}{r}\right) + \left(-\frac{r\sin^2\theta}{B}\right)\left(\frac{1}{r}\right) \\
& + (-\sin\theta\cos\theta)(\cot\theta) \\
=\ & -\frac{r\sin^2\theta A'}{2AB} - \frac{r\sin^2\theta B'}{2B^2} - \frac{2\sin^2\theta}{B} - \cos^2\theta
\end{aligned}
$$

$R_{\varphi\varphi}$ **Fourth Term** The non-zero Christoffel products involve the (r,φ) and (θ,φ) couplings.

$$
\begin{aligned}
\Gamma^{\lambda}_{\varphi\rho}\Gamma^{\rho}_{\lambda\varphi} =\ & \cancel{\Gamma^{t}_{\varphi t}\Gamma^{t}_{t\varphi}} + \cancel{\Gamma^{t}_{\varphi r}\Gamma^{r}_{t\varphi}} + \cancel{\Gamma^{t}_{\varphi\theta}\Gamma^{\theta}_{t\varphi}} + \cancel{\Gamma^{t}_{\varphi\varphi}\Gamma^{\varphi}_{t\varphi}} \\
& + \cancel{\Gamma^{r}_{\varphi t}\Gamma^{t}_{r\varphi}} + \cancel{\Gamma^{r}_{\varphi r}\Gamma^{r}_{r\varphi}} + \cancel{\Gamma^{r}_{\varphi\theta}\Gamma^{\theta}_{r\varphi}} + \Gamma^{r}_{\varphi\varphi}\Gamma^{\varphi}_{r\varphi} \\
& + \cancel{\Gamma^{\theta}_{\varphi t}\Gamma^{t}_{\theta\varphi}} + \cancel{\Gamma^{\theta}_{\varphi r}\Gamma^{r}_{\theta\varphi}} + \cancel{\Gamma^{\theta}_{\varphi\theta}\Gamma^{\theta}_{\theta\varphi}} + \Gamma^{\theta}_{\varphi\varphi}\Gamma^{\varphi}_{\theta\varphi} \\
& + \cancel{\Gamma^{\varphi}_{\varphi t}\Gamma^{t}_{\varphi\varphi}} + \Gamma^{\varphi}_{\varphi r}\Gamma^{r}_{\varphi\varphi} + \Gamma^{\varphi}_{\varphi\theta}\Gamma^{\theta}_{\varphi\varphi} + \cancel{\Gamma^{\varphi}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\varphi}} \\
=\ & \left(-\frac{r\sin^2\theta}{B}\right)\left(\frac{1}{r}\right) + (-\sin\theta\cos\theta)(\cot\theta) \\
& + \left(\frac{1}{r}\right)\left(-\frac{r\sin^2\theta}{B}\right) + (\cot\theta)(-\sin\theta\cos\theta) \\
=\ & -\frac{2\sin^2\theta}{B} - 2\cos^2\theta
\end{aligned}
$$

$R_{\varphi\varphi}$ Complete

$$
\begin{aligned}
R_{\varphi\varphi} = &\left(-\frac{\sin^2\theta}{B} + \frac{r\sin^2\theta B'}{B^2} + \sin^2\theta - \cos^2\theta\right) \\
&-0 \\
&+\left(-\frac{r\sin^2\theta A'}{2AB} - \frac{r\sin^2\theta B'}{2B^2} - \frac{2\sin^2\theta}{B} - \cos^2\theta\right) \\
&-\left(-\frac{2\sin^2\theta}{B} - 2\cos^2\theta\right) \\
&= -\frac{r\sin^2\theta A'}{2AB} + \frac{r\sin^2\theta B'}{2B^2} - \frac{\sin^2\theta}{B} + \sin^2\theta
\end{aligned}
$$

$$
R_{\varphi\varphi} = \sin^2\theta\left(-\frac{rA'}{2AB} + \frac{rB'}{2B^2} - \frac{1}{B} + 1\right) \tag{2.4}
$$

2.4.5. The Ricci Tensor

$$
R_{\mu\nu} = \begin{pmatrix}
-\frac{A''}{2B} + \frac{A'B'}{4B^2} + \frac{A'^2}{4AB} - \frac{A'}{rB} & 0 & 0 & 0 \\
0 & -\frac{A''}{2A} + \frac{A'B'}{4AB} + \frac{A'^2}{4A^2} + \frac{B'}{rB} & 0 & 0 \\
0 & 0 & -\frac{rA'}{2AB} + \frac{rB'}{2B^2} - \frac{1}{B} + 1 & 0 \\
0 & 0 & 0 & \sin^2\theta\left(-\frac{rA'}{2AB} + \frac{rB'}{2B^2} - \frac{1}{B} + 1\right)
\end{pmatrix}
$$

In the normal derivation of the Schwarzschild metric, at this point the fact that the Ricci Tensor is all zeros is used. However, I could not convince myself of the truth of that statement, so I decided to continue without it. Later in Section 2.5.6 I will show that it is.

2.4.6. The Ricci scalar

$$
\begin{aligned}
R = \quad & g^{\mu\nu} R_{\mu\nu} \\
= \quad & g^{tt} R_{tt} + g^{rr} R_{rr} + g^{\theta\theta} R_{\theta\theta} + g^{\varphi\varphi} R_{\varphi\varphi} \\
= \quad & \frac{1}{A}\left(-\frac{A''}{2B} + \frac{A'B'}{4B^2} + \frac{A'^2}{4AB} - \frac{A'}{rB}\right) \\
& + \frac{1}{B}\left(-\frac{A''}{2A} + \frac{A'B'}{4AB} + \frac{A'^2}{4A^2} + \frac{B'}{rB}\right) \\
& + \frac{1}{r^2}\left(-\frac{rA'}{2AB} + \frac{rB'}{2B^2} - \frac{1}{B} + 1\right) \\
& + \frac{1}{r^2 \sin^2\theta}\left(\sin^2\theta\left(\frac{rA'}{2AB} + \frac{rB'}{2B^2} - \frac{1}{B} + 1\right)\right) \\
= & -\frac{A''}{2AB} + \frac{A'B'}{4AB^2} + \frac{A'^2}{4A^2B} - \frac{A'}{rAB} \\
& -\frac{A''}{2AB} + \frac{A'B'}{4AB^2} + \frac{A'^2}{4A^2B} + \frac{B'}{rB^2} \\
& -\frac{A'}{2rAB} + \frac{B'}{2rB^2} - \frac{1}{r^2B} + \frac{1}{r^2} \\
& -\frac{A'}{2rAB} + \frac{B'}{2rB^2} - \frac{1}{r^2B} + \frac{1}{r^2} \\
= & -\frac{A''}{AB} + \frac{A'B'}{2AB^2} + \frac{A'^2}{2A^2B} - \frac{2A'}{rAB} + \frac{2B'}{rB^2} - \frac{2}{r^2B} + \frac{2}{r^2}
\end{aligned}
$$

2.5. The Einstein Field Equation

2.5.1. The tt Einstein Field Equation

$$R_{tt} - \frac{1}{2}g_{tt}R = 0$$

$$-\frac{A''}{2B} + \frac{A'B'}{4B^2} + \frac{A'^2}{4AB} - \frac{A'}{rB}$$
$$-\frac{1}{2}A\left(-\frac{A''}{AB} + \frac{A'B'}{2AB^2} + \frac{A'^2}{2A^2B} - \frac{2A'}{rAB} + \frac{2B'}{rB^2} - \frac{2}{r^2B} + \frac{2}{r^2}\right) = 0$$

distribute $-\frac{1}{2}A$

$$-\frac{A''}{2B} + \frac{A'B'}{4B^2} + \frac{A'^2}{4AB} - \frac{A'}{rB}$$
$$+\frac{A''}{2B} - \frac{A'B'}{4B^2} - \frac{A'^2}{4AB} + \frac{A'}{rB} - \frac{AB'}{rB^2} + \frac{A}{r^2B} - \frac{A}{r^2} = 0$$

cancel opposites

$$-\frac{AB'}{rB^2} + \frac{A}{r^2B} - \frac{A}{r^2} = 0$$

multiply by $-\frac{r}{A}$ and move the last term to the right hand side

$$\frac{B'}{B^2} - \frac{1}{rB} = -\frac{1}{r}$$

This is a Bernoulli differential equation. Substitute $V = -\frac{1}{B}$ and $V' = \frac{B'}{B^2}$

$$V' + \frac{V}{r} = -\frac{1}{r}$$

compute integrating factor, $e^{\int \frac{1}{r}\,\mathrm{d}r} = e^{\ln r} = r$, and multiply by it

$$rV' + V = -1$$
$$(rV)' = -1$$

integrate both sides

$$\int (rV)' = -\int 1\,\mathrm{d}r$$
$$rV = -r + C_1$$
$$V = -1 + \frac{C_1}{r}$$

recall $V = -\frac{1}{B}$ so $B = -\frac{1}{V}$

$$B = -\frac{1}{-1 + \frac{C_1}{r}}$$

$$\boxed{B = \frac{1}{1 - \frac{C_1}{r}}}$$

notice that this correctly matches the boundary condition $B(\infty) = \eta_{rr} = 1$

2.5.2. The rr Einstein Field Equation

$$R_{rr} - \frac{1}{2} g_{rr} R = 0$$

$$-\frac{A''}{2A} + \frac{A'B'}{4AB} + \frac{A'^2}{4A^2} + \frac{B'}{rB}$$
$$-\frac{1}{2}B\left(-\frac{A''}{AB} + \frac{A'B'}{2AB^2} + \frac{A'^2}{2A^2B} - \frac{2A'}{rAB} + \frac{2B'}{rB^2} - \frac{2}{r^2B} + \frac{2}{r^2}\right) = 0$$

distribute $-\frac{1}{2}B$

$$-\frac{A''}{2A} + \frac{A'B'}{4AB} + \frac{A'^2}{4A^2} + \frac{B'}{rB}$$
$$+\frac{A''}{2A} - \frac{A'B'}{4AB} - \frac{A'^2}{4A^2} + \frac{A'}{rA} - \frac{B'}{rB} + \frac{1}{r^2} - \frac{B}{r^2} = 0$$

cancel opposite terms

$$\frac{A'}{rA} + \frac{1}{r^2} - \frac{B}{r^2} = 0$$

multiply remaining terms by r and separate variables

$$\frac{A'}{A} = \frac{1}{r}B - \frac{1}{r}$$

substitute the value of B derived from the tt field equation

$$\frac{A'}{A} = \frac{1}{r}\frac{1}{1 - \frac{C_1}{r}} - \frac{1}{r}$$

simplify

$$\frac{A'}{A} = \frac{1}{r - C_1} - \frac{1}{r}$$

integrate both sides

$$\int \frac{A'}{A}\,\mathrm{d}r = \int \left(\frac{1}{r - C_1} - \frac{1}{r}\right)\mathrm{d}r$$

$$\ln|A| = \ln|r - C_1| - \ln(r) + C_2$$

raise both sides to e

$$e^{\ln|A|} = e^{\ln|r-C_1| - \ln(r)+C_2}$$

simplify

$$A = K(r - C_1)\frac{1}{r}$$

distribute $\frac{1}{r}$

$$A = K\left(1 - \frac{C_1}{r}\right)$$

apply boundary condition $A(\infty) = \eta_{tt} = -1$ to determine $K = -1$

$$\boxed{A = -\left(1 - \frac{C_1}{r}\right)}$$

2.5.3. Solve for C_1

A and B — and the relationship between them and a derivative.

$$A = -\left(1 - \frac{C_1}{r}\right)$$

$$B = \frac{1}{1 - \frac{C_1}{r}}$$

$$B = -\frac{1}{A}$$

$$A' = -\frac{C_1}{r^2}$$

Spacetime Coordinates and Proper Time

These are the spherical spacetime coordinates as functions of proper time τ. The t coordinate is multiplied by c so that all four coordinates will be in length units.

$$x^\mu = (ct(\tau), r(\tau), \theta(\tau), \varphi(\tau))$$

where τ is defined by the following relation

$$-c^2 \mathrm{d}\tau^2 \coloneqq \mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$$

For a non moving obj at some distance r from the center of a sphere the coordinates are

$$x^\mu = (ct(\tau), r, \theta, \varphi)$$

The Four-Velocity

The four-velocity is the ordinary derivative of the spacetime coordinates with respect to proper time.

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \left(\frac{\mathrm{d}(ct)}{\mathrm{d}\tau}, \frac{\mathrm{d}r}{\mathrm{d}\tau}, \frac{\mathrm{d}\theta}{\mathrm{d}\tau}, \frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)$$

For a motionless object this reduces to

$$u^\mu = \left(\frac{\mathrm{d}(ct)}{\mathrm{d}\tau}, 0, 0, 0\right)$$

Solve for $\frac{\mathrm{d}(ct)}{\mathrm{d}\tau}$ of a motionless object using the Schwarzschild metric

$$-c^2\mathrm{d}\tau^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$$

$$\begin{aligned}-c^2\mathrm{d}\tau^2 &= g^{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \\ &= g^{tt}\mathrm{d}x^t\mathrm{d}x^t + g^{rr}\mathrm{d}x^r\mathrm{d}x^r + g^{\theta\theta}\mathrm{d}x^\theta\mathrm{d}x^\theta + g^{\varphi\varphi}\mathrm{d}x^\varphi\mathrm{d}x^\varphi \\ &= g^{tt}\left(\mathrm{d}x^t\right)^2 + g^{rr}(\mathrm{d}x^r)^2 + g^{\theta\theta}\left(\mathrm{d}x^\theta\right)^2 + g^{\varphi\varphi}(\mathrm{d}x^\varphi)^2 \\ &= A(\mathrm{d}(ct))^2 + B\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\end{aligned}$$

divide both sides by $\mathrm{d}\tau^2$ and recall that in this case the positional derivatives are zero

$$-c^2 = A\left(\frac{\mathrm{d}(ct)}{\mathrm{d}\tau}\right)^2 + B\cancel{\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2} + r^2\cancel{\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2} + r^2\sin^2\theta\cancel{\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2}$$

simplify

$$\frac{\mathrm{d}(ct)}{\mathrm{d}\tau} = \sqrt{\frac{-c^2}{A}} = c\sqrt{-\frac{1}{A}}$$

recall $B = -\frac{1}{A}$

$$\frac{\mathrm{d}(ct)}{\mathrm{d}\tau} = c\sqrt{B} \tag{2.5}$$

so the four-velocity is

$$u^\mu = \left(c\sqrt{B}, 0, 0, 0\right)$$

As a check, verify the following invariant

$$-c^2 \stackrel{?}{=} u^\mu u_\mu$$

$$-c^2 \stackrel{?}{=} g_{tt}u^t u^t$$

$$-c^2 \stackrel{?}{=} A\left(c\sqrt{B}\right)\left(c\sqrt{B}\right)$$

$$-c^2 \stackrel{?}{=} Ac^2B$$

again recalling $B = -\frac{1}{A}$

$$-c^2 \equiv -c^2 \quad \checkmark$$

The Four-Acceleration

The four-acceleration is the Total Covariant Derivative with respect to proper time

$$a^{\mu} = \frac{\mathrm{D}u^{\mu}}{\mathrm{D}\tau} = \left(\frac{\mathrm{d}u^{\mu}}{\mathrm{d}\tau} + \Gamma^{\mu}_{\nu\rho}u^{\nu}u^{\rho}\right)$$

Compute the four-acceleration for a motionless object in Schwarzchild space. The only non-zero component will be a^r.

$$a^r = \frac{\mathrm{D}u^r}{\mathrm{D}\tau} = \left(\cancel{\frac{\mathrm{d}u^r}{\mathrm{d}\tau}} + \Gamma^{r}_{\nu\rho}u^{\nu}u^{\rho}\right)$$

$$a^r = \Gamma^r_{tt}u^t u^t$$

$$a^r = \frac{-A'}{2B}\left(c\sqrt{B}\right)^2$$

the Bs cancel

$$a^r = \frac{-c^2 A'}{2}$$

So the four-acceleration is:

$$a^{\mu} = \left(0, \frac{-c^2 A'}{2}, 0, 0\right)$$

The proper acceleration α is given by:

$$\alpha = \sqrt{a^{\mu}a_{\mu}} = \sqrt{g_{\mu\nu}a^{\mu}a^{\nu}}$$

For Schwarschild, since only a^r is non zero, this reduces to

$$\alpha = \sqrt{a^r a_r} = \sqrt{g_{rr}a^r a^r}$$

$$\alpha = a^r\sqrt{g_{rr}}$$

substitute in the equations for a^r and g_{rr}

$$\alpha = \frac{-c^2 A'}{2}\sqrt{B}$$

and then subsitute in the equations A' and B

$$\alpha = \frac{c^2 C_1}{2r^2}\sqrt{\frac{1}{1 - \frac{C_1}{r}}}$$

Newtons law of universal gravitation is

$$F = \frac{GMm}{r^2}$$

divide by m to get Newtonian gravitational acceleration

$$a = \frac{GM}{r^2}$$

Make Schwarschild GR proper acceleration approximate Newtonian gravitational acceleration.

$$\frac{GM}{r^2} \approx \frac{c^2 C_1}{2r^2}\sqrt{\frac{1}{1-\frac{C_1}{r}}}$$

Guess that $\frac{C_1}{r} \ll 1$

$$\frac{GM}{r^2} \approx \frac{c^2 C_1}{2r^2}\cancel{\sqrt{\frac{1}{1-\cancel{\frac{C_1}{r}}}}}$$

solve for C_1

$$C_1 \approx \frac{2GM}{c^2}$$

At the surface of the earth $\frac{C_1}{r} = 1.4 \times 10^{-9}$ which is $\ll 1$. And Actually, since G is measured and GR is the more accurate representations of reality, this is the exact value of C_1.

$$C_1 = \frac{2GM}{c^2}$$

Substituting C_1 into the equation for proper acceleration gives the exact GR equation.

$$\alpha = \frac{GM}{r^2}\sqrt{\frac{1}{1-\frac{2GM}{c^2 r}}}$$

Which means the Newtonian equation is the approximation.

$$a \approx \frac{GM}{r^2}$$

substitute C_1 into A and B

$$\boxed{A = -\left(1 - \frac{2GM}{c^2 r}\right)} \tag{2.6}$$

$$\boxed{B = \frac{1}{1-\frac{2GM}{c^2 r}}} \tag{2.7}$$

2.5.4. The Schwarzschild Metric

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2GM}{c^2 r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2GM}{c^2 r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

The Schwarzschild Radius r_s is defined as:

$$r_s := \frac{2GM}{c^2}$$

It is the key scaling parameter in Schwarzschild geometry. It is used to define the Metric function $f(r)$:

$$f(r) := 1 - \frac{r_s}{r}$$

With these defenitions the Schwarzschild Metric can be rewritten as:

$$g_{\mu\nu} = \begin{pmatrix} -f(r) & 0 & 0 & 0 \\ 0 & \frac{1}{f(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

2.5.5. The Christoffel Symbols

A and B and their derivatives can be written in terms of the Metric function f.

$$\begin{aligned} A &= -f \\ A' &= -f' \\ A'' &= -f'' B \ = f^{-1} \\ B' &= -f^{-2} f' \end{aligned}$$

Use the above to eliminate A and B from the Christoffel Symbols as derived in Section 2.3

$$\begin{aligned} \Gamma^t_{tr} = \Gamma^t_{rt} &= \frac{A'}{2A} = \frac{-f'}{2(-f)} = \frac{f'}{2} f^{-1} \\ \Gamma^r_{tt} &= \frac{-A'}{2B} = \frac{f'}{2f^{-1}} = \frac{f'}{2} f \\ \Gamma^r_{rr} &= \frac{B'}{2B} = \frac{-f^{-2}f'}{2f^{-1}} = -\frac{f'}{2} f^{-1} \\ \Gamma^r_{\theta\theta} &= -\frac{r}{B} = -\frac{r}{f^{-1}} = -rf \\ \Gamma^r_{\varphi\varphi} &= \frac{-r\sin^2\theta}{B} = \frac{-r\sin^2\theta}{f^{-1}} = -r\sin^2\theta f \end{aligned}$$

The Christoffel Symbols can now be written in terms of the Metric function, the Swarzchild Radius, or the Mass and fundemental contants. The Christoffel Symbols that do not depend on the Metric function have been restated here for completness.

$$\begin{aligned}
\Gamma^t_{tr} = \Gamma^t_{rt} &= \frac{f'}{2} f^{-1} = \frac{r_s}{2r^2}\left(1 - \frac{r_s}{r}\right)^{-1} = \frac{GM}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1} \\
\Gamma^r_{tt} &= \frac{f'}{2} f = \frac{r_s}{2r^2}\left(1 - \frac{r_s}{r}\right) = \frac{GM}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right) \\
\Gamma^r_{rr} &= -\frac{f'}{2} f^{-1} = -\frac{r_s}{2r^2}\left(1 - \frac{r_s}{r}\right)^{-1} = -\frac{GM}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1} \\
\Gamma^r_{\theta\theta} &= -rf = -r\left(1 - \frac{r_s}{r}\right) = -r\left(1 - \frac{2GM}{c^2 r}\right) \\
\Gamma^r_{\varphi\varphi} &= -r\sin^2\theta f = -r\sin^2\theta\left(1 - \frac{r_s}{r}\right) = -r\sin^2\theta\left(1 - \frac{2GM}{c^2 r}\right) \\
\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} &= \frac{1}{r} \\
\Gamma^\theta_{\varphi\varphi} &= -\sin\theta\cos\theta \\
\Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} &= \frac{1}{r} \\
\Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} &= \cot\theta
\end{aligned}$$

2.5.6. The Ricci Tensor is Zero

Previously, I noted that the Ricci tensor is zero, but I was not yet convinced of that fact.

Ricci Tensor R_{tt} — Start with the equation we derived previously:

$$R_{tt} = -\frac{A''}{2B} + \frac{A'B'}{4B^2} + \frac{A'^2}{4AB} - \frac{A'}{rB}$$

see Equation (2.1)

Replace A and B and their derivatives with f and its derivative.

$$R_{tt} = -\frac{-f''}{2f^{-1}} + \frac{(-f')(-f^{-2}f')}{4(f^{-1})^2} + \frac{-f'^2}{4(-f)(f^{-1})} - \frac{-f'}{r(f^{-1})}$$

Simplify

$$R_{tt} = \frac{f''}{2f^{-1}} - \frac{f'^2}{4} + \frac{f'^2}{4} + \frac{f'}{r(f^{-1})}$$

Elimate the middle terms and factor out an f

$$R_{tt} = \left(\frac{f''}{2} + \frac{f'}{r}\right)f$$

The first and second derivatives of the metric function are $f' = r_s/r^2$ and $f'' = -2r_s/r^3$. Substituting these into the above gives:

$$R_{tt} = \left(-\frac{r_s}{r^3} + \frac{r_s}{r^3}\right)f = 0$$

Ricci Tensor R_{rr} — Start with the equation we derived previously:

$$R_{rr} = -\frac{A''}{2A} + \frac{A'B'}{4AB} + \frac{A'^2}{4A^2} + \frac{B'}{rB}$$

see Equation (2.2)

Rewrite in terms of f and simplify

$$\begin{aligned}
R_{rr} &= -\frac{-f''}{2(-f)} + \frac{(-f')(-f^{-2}f')}{4(-f)(f^{-1})} + \frac{(-f')^2}{4(-f)^2} + \frac{-f^{-2}f'}{r(f^{-1})} \\
&= -\frac{f''}{2f} - \frac{f'^2}{4f^2} + \frac{f'^2}{4f^2} - \frac{f'}{rf} \\
&= -\left(\frac{f''}{2} + \frac{f'}{r}\right)\frac{1}{f} \\
&= \left(-\frac{r_s}{r^3} + \frac{r_s}{r^3}\right)\frac{1}{f} \\
&= -(0)\frac{1}{f} \\
&= 0
\end{aligned}$$

Ricci Tensor $R_{\theta\theta}$ — Start with the equation we derived previously:

$$R_{\theta\theta} = -\frac{rA'}{2AB} + \frac{rB'}{2B^2} - \frac{1}{B} + 1$$

see Equation (2.3)

Rewrite in terms of f and simplify

$$\begin{aligned}
R_{\theta\theta} &= -\frac{r(-f')}{2(-f)(f^{-1})} + \frac{r(-f^{-2}f')}{2(f^{-1})^2} - \frac{1}{f^{-1}} + 1 \\
&= -\frac{rf'}{2} - \frac{rf'}{2} - f + 1 \\
&= -rf' - f + 1 \\
&= -r\left(\frac{r_s}{r^2}\right) - \left(1 - \frac{r_s}{r}\right) + 1 \\
&= -\frac{r_s}{r} - 1 + \frac{r_s}{r} + 1 \\
&= 0
\end{aligned}$$

Ricci Tensor $R_{\varphi\varphi}$ — Start with the equation we derived previously:

$$R_{\varphi\varphi} = \sin^2\theta\left(-\frac{rA'}{2AB} + \frac{rB'}{2B^2} - \frac{1}{B} + 1\right)$$

see Equation (2.4)

Note that the term in parentheses equals $R_{\theta\theta}$:

$$\begin{aligned}
R_{\varphi\varphi} &= \sin^2\theta(R_{\theta\theta}) \\
&= \sin^2\theta(0) \\
&= 0
\end{aligned}$$

Ricci Tensor is Zero — The above have shown that for the Schwarzschild Metric all of the components of the Ricci Tensor are zero. Which also means that the Ricci scalar is zero.

3. The Schwarzschild Metric Applications

3.1. Time Dilation (Stationary)

Consider two stationary clocks in the Schwarzschild geometry, located at fixed radii r_1 and r_2:

$$x_1^\mu(\tau_1) = (ct_1(\tau_1), r_1, \theta_0, \varphi_0)$$
$$x_2^\mu(\tau_2) = (ct_2(\tau_2), r_2, \theta_0, \varphi_0)$$

Solve for $\frac{\mathrm{d}(ct)}{\mathrm{d}\tau}$ of a motionless object using the Schwarzschild metric

$$-c^2 \mathrm{d}\tau^2 = \mathrm{g}_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$$

$$\begin{aligned} -c^2 \mathrm{d}\tau^2 &= \mathrm{g}^{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= \mathrm{g}^{tt} \mathrm{d}x^t \mathrm{d}x^t + \mathrm{g}^{rr} \mathrm{d}x^r \mathrm{d}x^r + \mathrm{g}^{\theta\theta} \mathrm{d}x^\theta \mathrm{d}x^\theta + \mathrm{g}^{\varphi\varphi} \mathrm{d}x^\varphi \mathrm{d}x^\varphi \\ &= \mathrm{g}^{tt} (\mathrm{d}x^t)^2 + \mathrm{g}^{rr} (\mathrm{d}x^r)^2 + \mathrm{g}^{\theta\theta} (\mathrm{d}x^\theta)^2 + \mathrm{g}^{\varphi\varphi} (\mathrm{d}x^\varphi)^2 \\ &= -f(r)(\mathrm{d}(ct))^2 + f(r)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\varphi^2 \end{aligned}$$

Divide both sides by $\mathrm{d}\tau^2$ and recall that in this case the positional derivatives are zero

$$-c^2 = -f(r)\left(\frac{\mathrm{d}(ct)}{\mathrm{d}\tau}\right)^2 + f(r)^{-1}\cancel{\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2} + r^2\cancel{\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2} + r^2 \sin^2\theta\cancel{\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2}$$

$$\frac{\mathrm{d}(ct)}{\mathrm{d}\tau} = c\sqrt{f(r)^{-1}}$$

Divide both sides by the speed of light c to get the rate of change of coordinate time t with respect to proper time τ:

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{f(r)}}$$

Inverting provides the rate of change of proper time with respect to coordinate time:

$$\dot{\tau} = \frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{f(r)}$$

Given a coordinate time interval Δt, we can compute the proper time interval $\Delta\tau$:

$$\Delta\tau = \int_{t=t_0}^{t_0+\Delta t} \dot{\tau}\, \mathrm{d}t$$

Because in this case $\dot{\tau}$ is constant with respect to t:

$$\begin{aligned} &= \dot{\tau} \int_{t=t_0}^{t_0+\Delta t} \mathrm{d}t \\ &= \dot{\tau}\, |_{t_0}^{t_0+\Delta t} \\ &= \dot{\tau}\Delta t \end{aligned}$$

Compute the difference in elapsed proper times between two radii:

$$\begin{aligned}
\Delta\tau_{21} &:= \Delta\tau_2 - \Delta\tau_1 \\
&= \dot{\tau}_2 \Delta t - \dot{\tau}_1 \Delta t \\
&= (\dot{\tau}_2 - \dot{\tau}_1)\Delta t \\
&= (\dot{\tau}_2 - \dot{\tau}_1)\left(\Delta \frac{\tau_1}{\dot{\tau}_1}\right) \\
&= \left(\frac{\dot{\tau}_2}{\dot{\tau}_1} - 1\right)\Delta\tau_1 \\
&= \left(\frac{\sqrt{f(r_2)}}{\sqrt{f(r_1)}} - 1\right)\Delta\tau_1 \\
&= \left(\sqrt{\frac{f(r_2)}{f(r_1)}} - 1\right)\Delta\tau_1
\end{aligned}$$

$$\Delta\tau_{21} = \left(\sqrt{\frac{1 - \frac{r_s}{r_2}}{1 - \frac{r_s}{r_1}}} - 1\right)\Delta\tau_1$$

3.1.1. Earth

For Earth, the Schwarzschild radius is:

$$R_{s,\oplus} = \frac{2GM_\oplus}{c^2} = 8.870\text{ mm}$$

This would be the radius of the event horizon of a black hole with the mass of earth.

The following table shows the gravitational time dilation caused by the earth relative to a clock at sea level (r_1) These do not take into account the time dilation caused by the velocity of the clocks. It's as if the clocks are "hovering" at the given altitude. The next section will account for the velocity of the clocks.

Name	Alt (m)	$\dot{\tau}_2$	$\Delta\tau_{21}$ (1d)	$\Delta\tau_{21}$ (1y)
Dead Sea	−430	0.99999999930382	−4.0 ns	−1.4 µs
Death Valley	−86	0.99999999930386	−805.7 ps	−294.3 ns
Sea Level	0	0.99999999930387		
1 meter	1	0.99999999930387	19.1 ps	7.0 ns
Chandler, AZ	370	0.99999999930391	3.5 ns	1.2 µs
Mount Everest	8,848	0.99999999930483	83.4 ns	30.4 µs
Passenger Jet	10,000	0.99999999930496	94.2 ns	34.4 µs
ISS Orbit	409,000	0.99999999934586	3.6 µs	1.3 ms
GPS Orbit	20,189,000	0.99999999983301	45.7 µs	16.6 ms
Geosynchronous	35,793,000	0.99999999989481	51.0 µs	18.6 ms
Moon Orbit	378,029,000	0.99999999998846	59.1 µs	21.6 ms
Infinity	∞	1.00000000000000	60.1 µs	21.9 ms

Table 2: Earth Gravitational time dilation relative to Sea Level (r_1).

3.1.2. Sun

For the sun, the Schwarzschild radius is:

$$R_{s,\odot} = \frac{2GM_{\odot}}{c^2} = 3.0 \text{ km}$$

This table show the time dilation caused by the sun relative to a clock "hovering" at the radius of the earth's orbit (r_1).

Name	**Radius (au)**	$\dot{\tau}_2$	$\Delta\tau_{21}$ **(1d)**	$\Delta\tau_{21}$ **(1y)**
Sun Surface	0.0047	0.99999787743099	−182.5 ms	−1.1 min
Mercury	0.39	0.99999997450019	−1.3 ms	−493.2 ms
Venus	0.72	0.99999998635341	−326.2 µs	−119.1 ms
Earth	1.0	0.99999999012907		
Mars	1.5	0.99999999352165	293.1 µs	107.0 ms
Jupiter	5.2	0.99999999810335	688.9 µs	251.6 ms
Saturn	9.6	0.99999999896990	763.8 µs	278.9 ms
Uranus	19	0.99999999948592	808.4 µs	295.2 ms
Neptune	30	0.99999999967149	824.4 µs	301.1 ms
Infinity	∞	1.00000000000000	852.8 µs	311.5 ms

Table 3: Sun Gravitational time dilation relative to Earth Orbit (r_1)

3.1.3. Sagittarius A*

the Schwarzschild radius of Sagittarius A*[1], the Black hole at the center of the Milky is:

$$R_{\text{SgrA*}} = \frac{2GM_{\text{SgrA*}}}{c^2} = 12{,}690{,}000 \text{ km}$$

Name	**Radius**	$\dot{\tau}_2$	$\Delta\tau_{21}$ **(1d)**	$\Delta\tau_{21}$ **(1y)**
Event Horizon R_s	0.085 au	0.00000000000000	−1.0 d	−1.0 yr
$R_s + 1$ m	0.085 au	0.00000887688449	−23.9 hr	−365.2 d
Photon Sphere[2] $1.5R_s$	0.13 au	0.57735026918962	−10.1 hr	−154.3 d
ISCO[3] $3R_s$	0.25 au	0.81649658092772	−4.4 hr	−67.0 d
S2 Peribothron[4]	120 au	0.99964647605890	−30.5 s	−3.0 hr
S2 Apobothron	1,800 au	0.99997692281632	−1.9 s	−12.1 min
Earth	26,000 ly	0.99999999997420	−2.2 µs	−814.0 µs
Andromeda Galaxy	2,500,000 ly	0.99999999999973	−23.1 ns	−8.4 µs
Infinity	∞	1.00000000000000	0.0 s	0.0 s

Table 4: Sagittarius A* Gravitational time dilation relative to Infinity (r_1)

[1] A* is pronounced "A Star"
[2] The Radius that light orbits
[3] Innermost Stable Circular Orbit
[4] S2 is a star with a highly eccentric orbit around SgrA*. Bothron from the Ancient Greek βόθρος (bóthros), "pit."